22/9/22 **MATH4030 Tutorial**

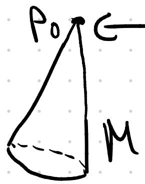
1) Assignment 1 due tonight @ 11:59pm on blackboard.

On the Regularity Condition

- Last time saw that in definition of a regular surface we had the regularity condition:

$\forall p, dX_p$ is full-rank $\iff X_u, X_v$ are linearly independent

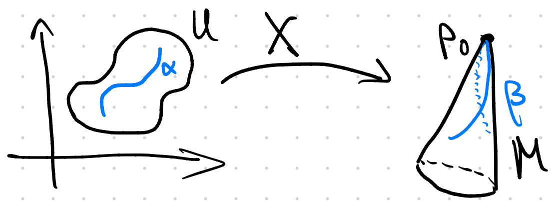
This is to avoid situations like:



Explain a little more how the regularity condition prevents situation above.

Recall from linear algebra that full-rank means

null-space of $dX = \{0\}$.



Suppose you have a regular curve α s.t. $X(\alpha(0)) = p_0$.

Then trace of β looks like

and $dX_{p_0}(\underbrace{\alpha'(0)}_{\neq 0}) = \beta'(0) = 0$ i.e. dX_{p_0} has a non-trivial null-space!

In other words, full-rank condition means tangent plane at each point has dimension > 0.

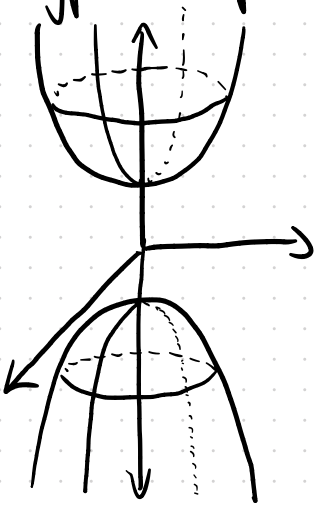
Regular Values and Inverse Images of Regular Values

Def: Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. $a \in \mathbb{R}$ is a regular value if $\forall x \in \mathbb{R}^3$ s.t. $f(x) = a$, $\nabla f(x) \neq 0$.

Otherwise, a is a critical value of f and x is a critical point.

Prop: Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and $a \in \mathbb{R}$ a regular value of f. Then $f^{-1}(a)$ is a regular surface.

Ex 1: Hyperboloid of 2 Sheets: $-x^2-y^2+z^2=1$. Show that this is a regular surface and find a parametrization.



Define $f(x,y,z) = -x^2-y^2+z^2-1$.

Clearly the surface is the inverse image

$$f^{-1}(0) = \{(x,y,z) : -x^2-y^2+z^2=1\}.$$

Clearly $f: \mathbb{R}^3 \to \mathbb{R}$ is smooth, and 0 is a regular value of f since

$\frac{\partial f}{\partial x} = -2x$, $\frac{\partial f}{\partial y} = -2y$, $\frac{\partial f}{\partial z} = 2z$. So ∇f vanishes only at $(0,0,0)$ and $(0,0,0) \notin f^{-1}(0)$. So it is a regular surface.

Rewrite as $x^2+y^2+1=z^2$. Taking $r^2=x^2+y^2$ (i.e. $x = r\cos v$, $y = r\sin v$)

Then we get $r^2+1=z^2 \iff 1=z^2-r^2$.

Then by hyperbolic trig identity $\cosh^2 u - \sinh^2 u = 1$ we take $z=\cosh u$, $r = \sinh u$, then we have

$$(x,y,z) = (\sinh u\cos v, \sinh u \sin v, \cosh u).$$

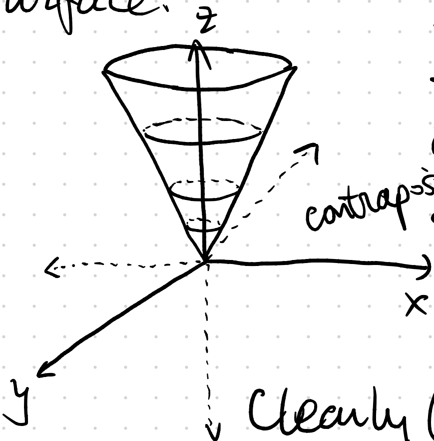
This is an example of a regular surface that is disconnected

Regular Surfaces are Graphs Locally

Prop: Let M be a regular surface and $X: U \to M$ be a local parametrization. Then for any $p = (u_0, v_0) \in U$, there is an open set $V \subset U$ with $p \in V$ s.t. $X(V)$ is the graph of a differentiable function in one of the coordinate planes.

i.e. $X(V) = (x, y, f(x,y))$ or $(x, g(x,z), z)$, $(h(y,z), y, z)$

Ex 2: One-sheeted cone $z = +\sqrt{x^2+y^2}$ is not a regular surface.



Pf: Use fact that regular surfaces are locally graphs

contrapositive $\Leftarrow$ there is a point where it cannot be a graph of a diff. f'n

$\Rightarrow$ not a regular surface.

Clearly $(0,0,0)$ is the problem point.

So let's show it is not the graph of a differentiable function locally around $(0,0,0)$. Suppose it is the graph of a diff. fn., then it could either be written $x = h(y,z)$, $y = g(x,z)$, $z = f(x,y)$.

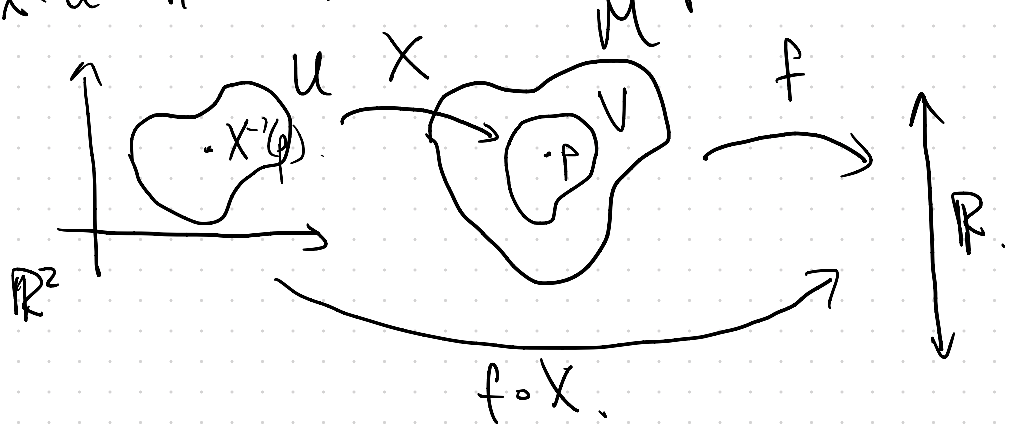
these ones not possible since projections onto xz, yz planes are not one-to-one, so the functions h, g would not be well-defined.

So examine $z = f(x,y)$, would have to take the form $z = +\sqrt{x^2+y^2}$ locally at 0.

But this function is not differentiable at $(0,0)$.

Differentiable Functions

<u>Def</u>: Let $f: V \subset M \to \mathbb{R}$ be a function defined in an open subset V of a regular surface M. f is differentiable at $p \in V$ if $\exists$ parametrization $X: U \subset \mathbb{R}^2 \to V \cap M$ with $p \in X(U)$ s.t. $f \circ X: U \to \mathbb{R}$ is differentiable at $X^{-1}(p)$.



<u>Ex</u>: The height function relative to a unit vector $v \in \mathbb{R}^3$ $h: M \to \mathbb{R}$ by $h(p) = p \cdot v$ is a differentiable function

Essentially because the dot product is differentiable.